2020A

Lecture 9

Fall 2020

- spherical coordinates
- change of variables.

when the domain of integration is of the form

$$\Omega = \{(x,y,z) : (x,y) \in D,\ f_1(x,y) \leq z \leq f_2(x,y)\}$$

we have

$$\iiint_{\Omega} F = \iint_{D} \int_{f_1(x,y)}^{f_2(x,y)} F(x,y,z)\,dz\,dA(x,y).$$

When D can be described by polar coordinates

$$D = \{(x,y) : x = r\cos\theta,\ y = r\sin\theta$$
$$\theta_1 \leq \theta \leq \theta_2,\ r_1(\theta) \leq r \leq r_2(\theta)\}$$

we introduce cylindrical coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

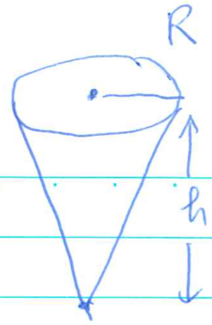
so

$$\iiint_{\Omega} F = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{\tilde{f}_1(r,\theta)}^{\tilde{f}_2(r,\theta)} \tilde{F}(r,\theta,z)\,dz\,r\,dr\,d\theta,$$

when

$$\tilde{F}(r,\theta,z) = F(r\cos\theta, r\sin\theta, z)$$
$$\tilde{f}_i(r,\theta) = f_i(x,y), \quad i = 1,2.$$

e.g. 7. Find the volume of



In e.g. 5 we used cross section method to do it. Now, we use cylindrical coordinates.

$\frac{h}{R}\sqrt{x^2+y^2} \leq z \leq h$ is converted to

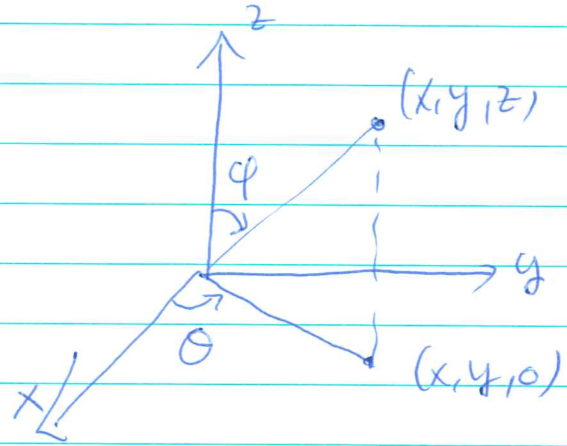
$$\underbrace{\frac{h}{R} r}_{\tilde{f}_1} \leq z \leq \underbrace{h}_{\tilde{f}_2},$$

$$0 \leq r \leq R$$
$$0 \leq \theta \leq 2\pi$$

∴ Let C be the cone

$$|C| = \iiint_C 1 \, dV$$
$$= \int_0^{2\pi} \int_0^R \int_{\frac{h}{R} r}^{h} 1 \, dz \, r \, dr \, d\theta$$
$$= 2\pi \int_0^R \left(h - \frac{h}{R} r\right) r \, dr$$
$$= 2\pi \left(h \frac{r^2}{2} - \frac{h}{R} \frac{r^3}{3} \right) \Big|_0^R$$
$$= \frac{1}{3} \pi R^2 h. \ \#$$

Now, a point (x,y,z) can also be described by spherical coordinates (ρ, φ, θ) where ρ is the distance (x,y,z) to $(0,0,0)$, φ the angle it makes with z-axis, and θ is the polar one. The relations are



$$x = \rho \sin\varphi \cos\theta$$
$$y = \rho \sin\varphi \sin\theta$$
$$z = \rho \cos\varphi ,$$

$$0 \le \varphi \le \pi, \quad 0 \le \theta < \pi, \quad \rho \ge 0 .$$

When Ω can be described as

$$\Omega = \{ (x,y,z) : \rho_1(\varphi,\theta) \le \rho \le \rho_2(\varphi,\theta), \ \varphi_1 \le \varphi \le \varphi_2, \ \theta_1 \le \theta \le \theta_2 \}$$

we may use spherical coordinates to do integration.

<u>Theorem</u> 4 Let F be conti in Ω (described above).

$$\iiint_{\Omega} F(x,y,z)\, dV = \int_{\theta_1}^{\theta_2} \int_{\varphi_1}^{\varphi_2} \int_{\rho_1(\varphi,\theta)}^{\rho_2(\varphi,\theta)} \tilde{F}(\rho,\varphi,\theta)\, \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

where $\tilde{F}(\rho,\varphi,\theta) = F(\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi)$.

We refer the pf to Text 15.7, difficult to draw the pictures!

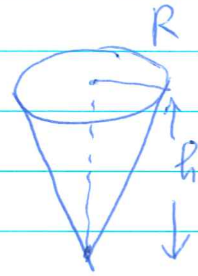
In spherical coordinate, equations look very different.

- $\rho = c$ describes the sphere of radius c, center $(0,0,0)$, for

$\sqrt{x^2+y^2+z^2} = c$, ie $x^2+y^2+z^2 = c^2$.

- $\theta = c$ is a plane $\perp$ to the xy-plane. For example, $x - y = 0$ is $\rho \sin\varphi \cos\theta - \rho \sin\varphi \sin\theta = 0$, ie $\tan\theta = 1$, so $\theta = \pi/4$.
- $\varphi = c$ is a circular cone. For example, $z = \sqrt{x^2+y^2}$, ie, $\rho\cos\varphi = \sqrt{\rho^2\cos^2\theta \sin^2\varphi + \rho^2 \sin^2\theta \sin^2\varphi}$, $\rho\cos\varphi = \rho\sin\varphi$, ie $\tan\varphi = 1$ or $\varphi = \pi/4$.

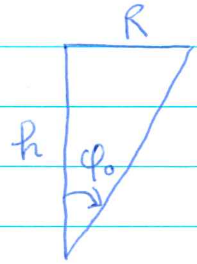
e.g 8 Use spherical coordinate to find the volume of



$$\Omega = \{(x,y,z) : \frac{h}{R}\sqrt{x^2+y^2} \le z \le h,\ x^2+y^2 \le R^2\}$$

$$\tilde{\Omega} = \{(\rho,\varphi,\theta) : 0 \le \rho \le \rho_2(\varphi,\theta),\ 0 \le \varphi \le \varphi_0,\ 0 \le \theta \le 2\pi\}$$ where

$\rho_2(\varphi,\theta)$ describe $z = h$ and φ_0 is



$z = h \iff \rho\cos\varphi = h$

$$\tan\varphi_0 = \frac{R}{h}$$

$$\therefore \rho_2 = \frac{h}{\cos\varphi}.$$

$$\therefore |\Omega| = \int_0^{2\pi}\int_0^{\varphi_0}\int_0^{\frac{h}{\cos\varphi}} 1\ \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= 2\pi \int_0^{\varphi_0} \frac{1}{3}\frac{h^3}{\cos^3\varphi} \sin\varphi\, d\varphi$$

$$\textcircled{t} \quad \frac{1}{2}\frac{1}{t^2}$$

$$t = \cos\varphi$$

$$= \frac{2\pi}{3}\int_0^{\varphi_0} \frac{h^3}{\cos^3\varphi}\sin\varphi\, d\varphi. \qquad \frac{dt}{d\varphi} = -\sin\varphi.$$

$$= \frac{2\pi}{3}\int_1^{t_0} \frac{h^3}{t^3}(-1)\,dt \quad , \quad t_0 = \cos\varphi_0$$

$$\tan\varphi_0 = \frac{R}{h}$$

$$= \frac{2\pi}{3}h^3\frac{1}{2}\frac{1}{t^2}\Big|_1^{t_0} \qquad 1+\tan^2\varphi_0 = \sec^2\varphi_0$$

$$= 1+\frac{R^2}{h^2}$$

$$= \frac{2\pi}{3}h^3\frac{1}{2}\left(\frac{1}{t_0^2}-1\right) \qquad \therefore \frac{1}{t_0^2} = 1+\frac{R^2}{h^2}$$

$$= \frac{\pi}{3}h^3\frac{R^2}{h^2} = \frac{1}{3}\pi R^2 h \ . \ \#$$

e.g. 9. Evaluate $\displaystyle\iiint_\Omega \frac{x^2+y^2}{\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz$ where

Ω is the 1st octant of the ball: $x^2+y^2+z^2 \le 1$.

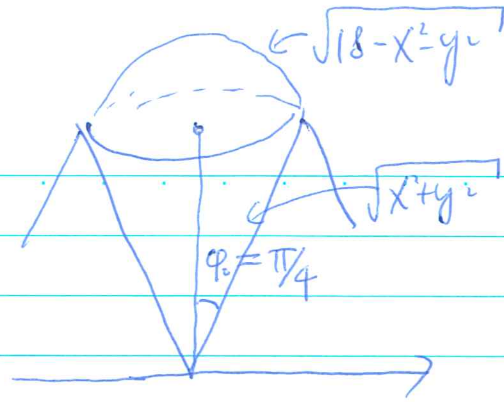
Clearly $\tilde{\Omega} = \{(r,\varphi,\theta) : 0 \le r \le 1,\ 0 \le \varphi \le \frac{\pi}{2},\ 0 \le \theta \le \frac{\pi}{2}\}$

$$\therefore \quad I = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{\rho^2\sin^2\varphi}{\rho}\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$= \frac{\pi}{2}\int_0^{\pi/2}\frac{\sin^4\varphi}{4}\sin\varphi\,d\varphi = \cdots \quad \#$$

e.g. 10 Express $\displaystyle\int_0^3\int_0^{\sqrt{9-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\,dz\,dx\,dy$

in cylindrical and spherical coordinates.



In cylindrical coordinate.

$$\{(r,\theta,z) : \sqrt{x^2+y^2} \le z \le \sqrt{18-x^2-y^2},\ 0 \le r \le \sqrt{18},\ 0 \le \theta \le 2\pi\}$$

$$\therefore\ I = \int_0^{2\pi}\int_0^{\sqrt{18}}\int_r^{\sqrt{18-r^2}} (r^2+z^2)\,dz\, r\,dr\,d\theta$$

In spherical coordinates

$$\{(\rho,\varphi,\theta) : 0 \le \rho \le \rho_2(\varphi,\theta),\ 0 \le \varphi \le \pi/4,\ 0 \le \theta \le 2\pi\}$$

when ρ_2 describes the sphere $\sqrt{18-x^2-y^2}$, so $\rho_2(\varphi,\theta) = \sqrt{18}$ a constant.

$$I = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{18}} \rho^2\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta.$$

eg 11 Find the volume bdd by the curves

$$(x^2+y^2+z^2)^2 = a^2(x^2+y^2-z^2).$$

Use spherical coordinate,

$$\rho^4 = -a^2\rho^2\cos 2\varphi \quad , \text{ or}$$

$$\rho^2 = -a^2\cos 2\varphi. \quad , 0 \le \varphi \le \pi.$$

Need $\cos 2\varphi \le 0$ to get solution, $2\varphi \in [\frac{\pi}{2}, \frac{3\pi}{2}]$

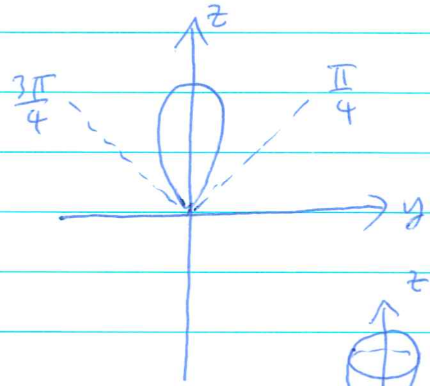
$\Longleftrightarrow \varphi \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

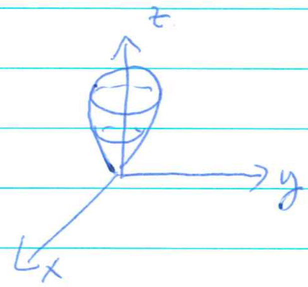
The solid is obtained by rotating this curve around the z-axis.

$0 \leq \rho \leq \sqrt{-a^2 \cos 2\varphi}$

$\frac{\pi}{4} \leq \varphi \leq \frac{3\pi}{4}$

$0 \leq \theta \leq 2\pi$





$$\text{Volume} = \int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \int_0^{\sqrt{-a^2\cos 2\varphi}} 1 \times \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$= \cdots$ #

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Change of Variables

Recall 1-dim case.

<u>Theorem 1</u> (Version 1) Let $\varphi : [\alpha, \beta] \to \mathbb{R}$ be C^1. For conti f,

$$\int_\alpha^\beta f(\varphi(t))\varphi'(t)\,dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx.$$

Pf. Let F be the anti-derivative of f, ie, $F' = f$.

$$\frac{d}{dt} F(\varphi(t)) = F'(\varphi(t))\varphi'(t) \quad \text{(Chain rule)}$$
$$= f(\varphi(t))\varphi'(t)$$

$$\therefore \quad F(\varphi(t))\Big|_{\alpha}^{\beta} = \int_{\alpha}^{\beta} F'(\varphi(t))\,dt \quad \text{(fundamental thm of Calculus)}$$

$$= \int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t)\,dt.$$

On the other hand,

$$F(\varphi(t))\Big|_{\alpha}^{\beta} = F(\varphi(\beta)) - F(\varphi(\alpha))$$

$$= \int_{\varphi(\alpha)}^{\varphi(\beta)} F'(x)\,dx \quad \text{(fundamental thm of Calculus)}$$

$$= \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx. \quad \#$$

Theorem 2 (Version II) Suppose further φ is 1-1, maps $[\alpha, \beta]$ onto $[a, b]$. then

$$\int_{\alpha}^{\beta} f(\varphi(t))\,|\varphi'(t)|\,dt = \int_{a}^{b} f(x)\,dx$$

Pf: φ is 1-1, so (a) $\varphi' \geq 0$, $\varphi(\alpha) = a$, $\varphi(\beta) = b$, or

(b) $\varphi' \leq 0$, $\varphi(\alpha) = b$, $\varphi(\beta) = a$.

In the 1st case $|\varphi'(t)| = \varphi'(t)$, so

$$\int_{\alpha}^{\beta} f(\varphi(t))|\varphi'(t)|\,dt = \int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t)\,dt$$

$$= \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx$$

$$= \int_a^b f(x)\,dx .$$

In the 2nd case, $\varphi'(t) \le 0$. so $-|\varphi'(t)| = \varphi'(t)$.

$$\int_\alpha^\beta f(\varphi(t))\,|\varphi'(t)|\,dt = -\int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt$$

$$= -\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx$$

$$= -\int_b^a f(x)\,dx$$

$$= \int_a^b f(x)\,dx . \quad \#$$